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Packet Radio Temporary Note #49

THE SPATIAL CAPACITY OF AN ALOHA CHANNEL

1. INTRODUCTION

Consider a single radio receiver, receiving packets transmitted at random times from a large number of terminals distributed about a central receiver in some fashion. In the usual analysis of an ALOHA channel it is assumed that if two packets overlap, neither packet is received correctly and both must be retransmitted. In ARPANET Satellite System Note 8 [1], Roberts introduced the idea of capture ratio to allow for the capture effect of FM receivers where the stronger of two overlapping packets may be received without error and only the weaker packet needs to be retransmitted.

In this note we analyze the reception of packets in the presence of capture effect. We show that when the central receiver is surrounded by a uniform density population of packet transmitting terminals, there exists a radius r_0 such that any packet transmitted from a terminal located within r_0 will be received correctly with probability one (after a finite number of retransmissions) while the expected number of retransmissions required for a packet transmitted from a terminal further from the center than r_0 will be unbounded. Thus there exists a natural circle of radius r_0 such that terminals transmitting from within this circle can get their packets into the central receiver, while terminals transmitting from outside this circle spend all of their time retransmitting their packets in vain. We call r_0 the Sisyphus distance of the ALOHA channel.

Furthermore we show that the total normalized packet throughput of all terminals in the circle defined by the Sisyphus distance is 1/2.

2. THE THEORY

Define S(r) as the normalized packet throughput per unit area, at a distance r. (As usual throughput is the product of the average number of successful packet transmissions per unit time and the packet length.) Then the throughput due to all the terminals in a differential ring of width dr

at a radius r is

$$S(r)\ 2\pi r\ dr$$

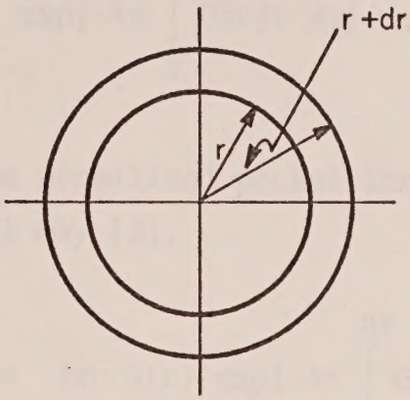


Figure 1 DIFFERENTIAL RING AT RADIUS r

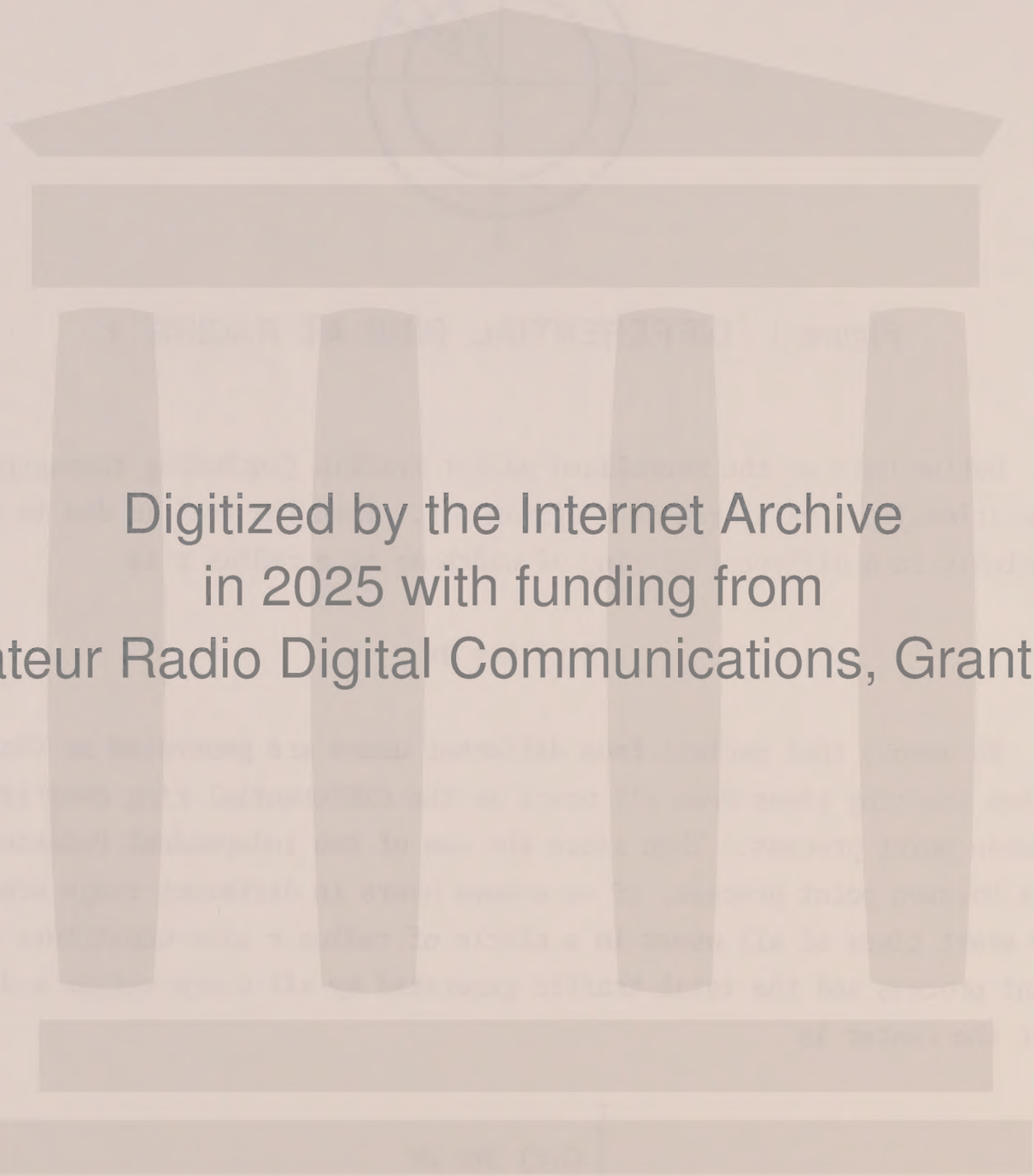
Define G(r) as the normalized packet traffic (including throughput plus repetitions) per unit area at a distance r. Then the traffic due to all the terminals in a differential ring of width dr at a radius r is

$$G(r)\ 2\pi r\ dr$$

We assume that packets from different users are generated so that the packet starting times from all users in the differential ring constitute a Poisson point process. Then since the sum of two independent Poisson processes is a Poisson point process, if we assume users in different rings are independent the start times of all users in a circle of radius r also constitute a Poisson point process and the total traffic generated by all users within a distance r of the center is

$$\int_0^r G(r)\ 2\pi r\ dr \qquad (1)$$

If we assume that a packet from a user at a distance r from the center will be received correctly unless it is overlapped by a packet sent from a user at a distance ar or less ($a \geq 1$) the probability that such a packet will be



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received correctly is [2]

$$\exp[-4\pi \int_0^{ar} G(r)r\, dr] \quad . \tag{2}$$

Then we can relate the normalized packet throughput to the normalized packet traffic in the usual way [2].

$$2\pi r\, S(r)\, dr = 2\pi r\, G(r)\, \exp[-4\pi \int_0^{ar} G(r)r\, dr]\, dr$$

or

$$S(r) = G(r)\, \exp[-4\pi \int_0^{ar} G(r)r\, dr] \tag{3}$$

If we now take a derivative of (3) with respect to r and then use (3) to substitute for the exponential, we get

$$S'(r)G(r) = G'(r)S(r) - 4\pi r a^2 S(r)G(r)G(ar) \tag{4}$$

Now we assume the packet throughput per unit area is a constant, independent of r

$$S(r) \triangleq S_0 \tag{5}$$

Then (4) becomes

$$G'(r) = 4\pi r a^2 G(r)G(ar) \tag{6}$$

For the case of a=1 (i.e., the case of a capture ratio of 0 db or perfect capture), (6) becomes

$$G'(r) = 4\pi r\, G^2(r) \tag{7}$$

with the boundary condition

$$G(0) = S_0 \tag{8}$$

Upon integrating (7) and using (8), we get

$$G(r) = \frac{S_0}{1-2\pi r^2 S_0} \tag{9}$$

Note that the normalized packet traffic per unit area is finite for

$$0 \leq r < r_0 \tag{10}$$

where

$$r_0 \triangleq [2\pi S_0]^{-1/2} \tag{11}$$

and r_0 is the Sisyphus distance mentioned in the introduction. Note that the Sisyphus distance also has the property that

$$\boxed{\pi r_0^2 S_0 = \frac{1}{2}} \tag{12}$$

That is, the spatial channel capacity of this ALOHA channel with perfect capture is 1/2. The channel will receive at a maximum total throughput equal to 1/2 the rate of a conventional channel.

If we rewrite (9) in terms of the Sisyphus distance we obtain

$$\frac{G(r)}{S_0} = \frac{1}{1-\left(\frac{r}{r_0}\right)^2} \tag{13}$$

and the ratio on the left of (13) is the average number of repetitions needed to transmit a packet from a terminal at a distance r from the center. This quantity is plotted in Figure 2.

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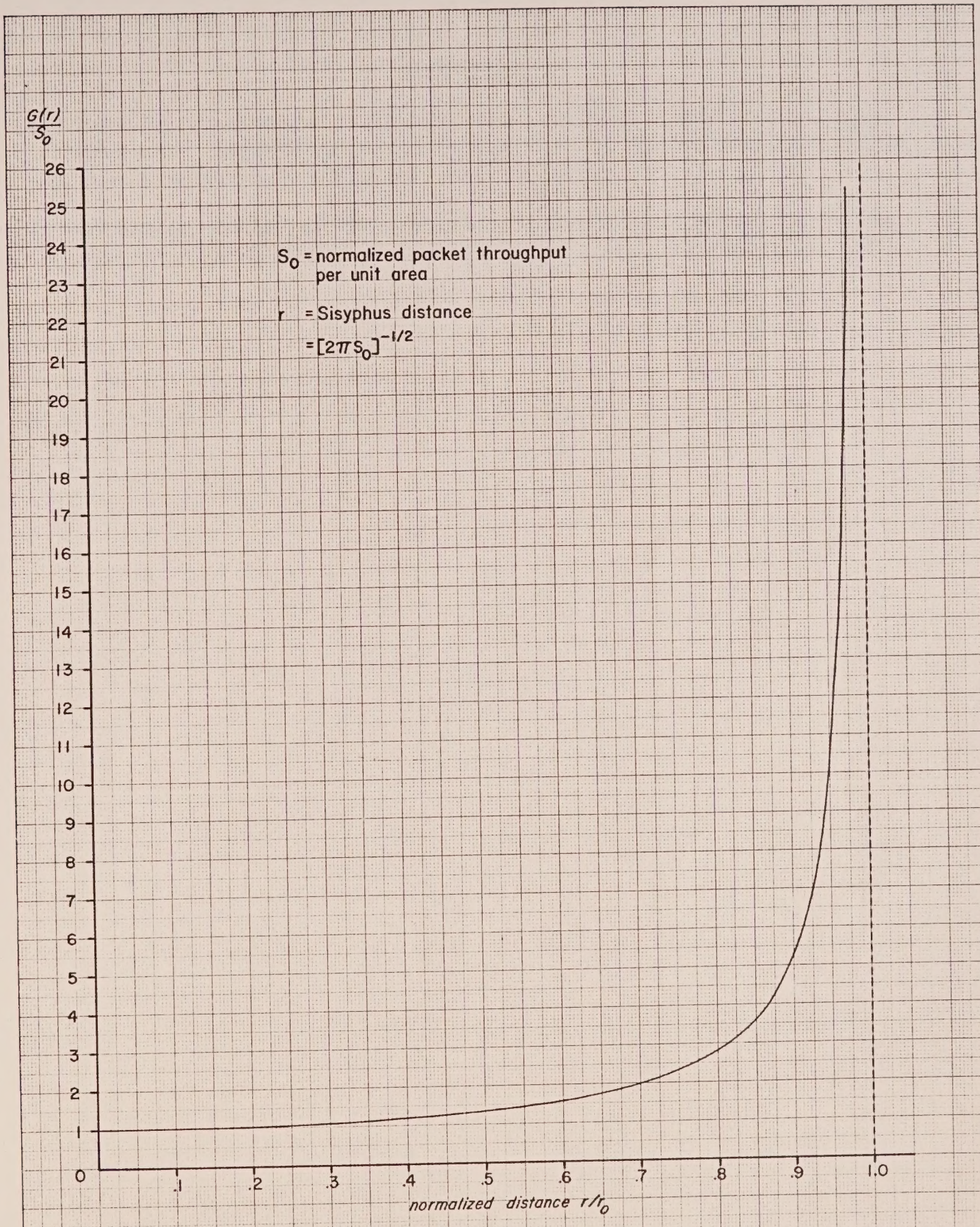

Figure 2 $G(r)/S_0$, THE AVERAGE NUMBER OF TRANSMISSIONS PER SUCCESSFUL TRANSMISSION AT A DISTANCE r

REFERENCES

[1] Roberts, Lawrence G., "ALOHA Packet System With and Without Slots and Capture," *ARPANET Satellite System Note 8, NIC Document No. 11290*, Stanford Research Institute, June 26, 1972.

[2] Abramson, Norman and Kuo, Franklin F., Editors, "THE ALOHA SYSTEM," *Computer-Communication Networks* (Chapter 14), Prentice-Hall, 1973.